Maximum Principle for Tensors

The version discussed now is applicable to 2-tensors, curvature 4-tensors and more general tensors satisfying heat-type equations.

The Set-up

Let $\pi : V \longrightarrow M^n$ be a rank r real vector bundle w/ M^n closed.

and having a time-dependent metric $g(t)$.

Let h be a fixed bundle metric on V. and $\nabla(t)$ be a time-dependent connections compatible w/ h, i.e.,

$$X(h(u,v)) = h(\nabla(t)_X u, v) + h(u, \nabla(t)_X v) \quad \forall\ X \in \Gamma(TM),\ u,v \in \Gamma(V).$$

The time-dependent Laplacian is

$$\Delta = \mathrm{tr}_g (\nabla^D \circ \nabla) \quad \text{w/}$$

$\nabla^D(t) : \Gamma(V \otimes T^*M) \longrightarrow \Gamma(V \otimes T^*M \otimes T^*M)$ is defined using the connection $\nabla(t)$ and the Levi-Civita connection $D(t)$ w.r.t $g(t)$.

$$\nabla^D(t)_X (u \otimes \alpha) = (\nabla(t)_X u) \otimes \alpha + u \otimes (D(t)_X \alpha).$$

i.e., ∇^D is just a connection on $V \otimes T^*M$ defined by the "product rule".

so $\nabla^D \circ \nabla : \Gamma(V) \longrightarrow \Gamma(V \otimes T^*M) \longrightarrow \Gamma(V \otimes T^*M \otimes T^*M)$.

Let $p \in M$ and $t \in [0,T)$, we can express the Laplacian at (p,t) as follows.

If $\gamma(\sigma)$ is a path in M then $v(\sigma) \in \Gamma(V_{\gamma(\sigma)})$ is parallel along

γ if

$$\nabla(t)_{\dot\gamma} v = 0.$$

$\forall$ path $\gamma : [0,b] \longrightarrow M$ and $v \in V_{\gamma(0)}$ $\exists!$ parallel section $v(\sigma) \in \Gamma(V_{\gamma(\sigma)})$

$\sigma \in [0,b]$, along γ w/ $v(0) = v$.

So if $s(p) \in V_p$ is a given vector, we can extend $s(p)$ to a section s of V over a small nbd $U \ni p$ by parallel translating $s(p)$ along any geodesic γ emanating from p.

For any section $u \in V$ we can write $u = \sum_{\alpha=1}^{r} u^\alpha s_\alpha$ w/ $\{s_\alpha\}$ a basis of local sections which are obtained from parallel transport. So $\Delta u(p) = \sum (\Delta u^\alpha)(p) \cdot s_\alpha(p)$.

So we can talk about the heat-type equations for sections of V.

for a family of sections $u(t) \in \Gamma(V)$, it satisfies a heat-type equation if

$$\frac{\partial u}{\partial t} = \Delta u + \nabla_{X(t)} u + F(u,t)$$

where $X(t)$ is a time-dependent v.f. on M and $F : V \times [0,T) \longrightarrow V$ is a fiber-preserving map.

just like the case of scalars, we consider the system of ODE on the fiber $\mathcal{V}_x = \pi^{-1}(x)$ corresponding to the eq^n. above as

$$\frac{dU}{dt} = F_x(U,t) \quad \text{where } F_x : \mathcal{V}_x \times [0,T] \longrightarrow \mathcal{V}_x.$$

<u>Def^n</u> (invariance under parallel translation) Let $P \subset \mathcal{V}$ be a subset and $P_x = P \cap V_x$ for $x \in M$. We say that P is invariant under parallel translation if $\forall$ path $\gamma : [0,b] \to M$ and vector $v \in P_{\gamma(0)}$, the unique parallel section $v(\sigma) \in \mathcal{V}_{\gamma(\sigma)}$, $\sigma \in [0,b]$ along $\gamma(\sigma)$ w/ $v(0) = v$ is contained in P.

<u>Thm.</u> (Max. principle for tensors)

Let $K \subset \mathcal{V}$ be a closed subset of $\mathcal{V}$ which is invariant under parallel translations w.r.t. $\nabla(t)$, $t \in [0,T)$ and convex in the fibres, i.e. K_x is convex $\forall x \in M$. Suppose $F(u,t)$ is continuous in (u,t) and is locally Lipschitz in u. Suppose that K has the property that for any $t_0 \in [0,T)$ and $U(t_0) \in K_x$, the solution $U(t)$, $t \in [t_0, T')$ to the ODE

$$\frac{dU}{dt} = F_x(U,t),$$

remains in K_x. Then any solution $u(x,t)$, defined for $x \in M$ and $t \in [0,T)$ to the PDE

$$\frac{\partial u}{\partial t} = \Delta u + \nabla_{X(t)} u + F(u,t)$$

which starts in $\mathcal{K}$ at $t=0$ (i.e. $u(x,0) \in K_x \ \forall x \in M$) remains in $\mathcal{K}$ $\forall\, t \in [0,T)$.

Remark:- intersection of two closed, invariant under parallel trans. fiberwise convex sets $\mathcal{K}_1$ and $\mathcal{K}_2$ is closed, inv. under. $\parallel^r$ translation and fiberwise convex.

Understanding the assumptions of the theorem

Suppose $\mathcal{V}$ is the trivial line bundle $M \times \mathbb{R} \to M$.

$\Rightarrow$ the inv. under parallel translation of $\mathcal{K}$ corresponds to the interval $[C_1, C_2]$ being independent of $x \in M$.

Convexity in fibres $\rightsquigarrow$ $[C_1, C_2]$ convex in $\mathbb{R}$.

In the following, it'll be useful to keep in mind the case when $u = Rm : \Lambda^2 T^*M \to \Lambda^2 T^*M$ and Rm is viewed as a self-adjoint operator on $\Lambda^2 T^*M$. So, e.g. below $V = \Lambda^2 T^*M \otimes_S \Lambda^2 T^*M$.

In all our applications: the v.b. will be of the form $V = W \otimes_S W$ w/ W another v.b. Using the metric $W^* \cong W$. $\Rightarrow V = \mathrm{End}_{\text{self-adjoint}}(W)$.

now suppose that $u \in V_p$ for some $p \in M$ and

$$\gamma : [0,1] \longrightarrow M \quad w/ \ \gamma(0) = p.$$

If $\tilde{u}$ is the unique parallel lift of γ s.t. $\tilde{u}(0) = u$.

Suppose $w \in W_p$ is an eigensection of u, i.e, $\exists\ \lambda \in \mathbb{R}$

$$u(w) = \lambda w.$$

Let $\tilde{w}$ be the unique parallel lift of w w/ $\tilde{w}(0) = w$.

Claim:- $\tilde{w}$ is an eigensection of $\tilde{u}$ with the same eigenvalue λ.

proof $\nabla_{\dot{\gamma}}(\tilde{u}(\tilde{w}) - \lambda\tilde{w}) = 0 \Rightarrow \tilde{u}(\tilde{w}) - \lambda\tilde{w} =$ constant along γ

and at 0, $\tilde{u}(\tilde{w})(0) = \lambda\tilde{w}(0)$.

So, if $r = \mathrm{rank}(W)$ and $u \in V_p$, let

$$\lambda_1(u) \geq \dots \geq \lambda_r(u) \text{ be the ordered eigenvalues of } u.$$

Consider the set

$$\Gamma = \{ (\lambda_1, \dots, \lambda_r) \in \mathbb{R}^r \mid \lambda_1 \geq \dots \geq \lambda_r \}.$$

If $w_1, \dots, w_r \in W_p$ are unit eigensections of $u \in V_p$ corresponding to $\lambda_1 \geq \dots \geq \lambda_r$, i.e, $u = \sum_{a=1}^{r} \lambda_a w_a \otimes w_a$, then given any path $\gamma : [0,1] \longrightarrow M$ w/ $\gamma(0) = p$, let $\tilde{w}_a : [0,1] \to W$ be the unique parallel lift of w_a w/ $\tilde{w}_a(0) = w_a$. Then

$$\tilde{u} = \sum_{a=1}^{r} \lambda_a \tilde{w}_a \otimes \tilde{w}_a$$

is a parallel lift of u w/ $\tilde{u}(0) = u$.

$\Rightarrow \quad \lambda_a(\tilde{u}) = \lambda_a(u) \quad \forall \; a = 1, \dots, r.$

Lemma (Criterion for invariance under parallel translation)

Suppose $G : \Gamma \longrightarrow \mathbb{R}$ is a function. If $c \in \mathbb{R}$ and

$$K = \{ u \in \mathcal{V} \mid G(\lambda_1(u), \dots, \lambda_r(u)) \leq c \}.$$

Then the subset $K \subset \mathcal{V}$ is invariant under parallel translation.

for convexity of sets like K: we just show that

$u \longmapsto G(\lambda_1(u), \dots, \lambda_r(u))$ is a convex function.

e.g. The subset $K = \{ u \in \mathcal{V} \mid \lambda_a(u) \geq 0 \;\; \forall \; a = 1, \dots, r \}$, i.e., the core of non-negative tensors is invariant under parallel transla-tion.

Remark :- sum of convex functions is convex. If f is convex then αf is convex if $\alpha > 0$, f convex. so is f^p, $p \in (1, \infty)$.

main application

$$Rm : \Lambda^2 T^*M \longrightarrow \Lambda^2 T^*M \Rightarrow Rm \in \Gamma(V), \; V = Sym^2(\Lambda^2 M).$$

$$\partial_t Rm = \Delta Rm + Rm^2 + Rm^{\#}.$$

The associated ODE is

$$\frac{d}{dt} M = M^2 + M^{\#}.$$

Thm. (Hamilton) If $(M^n, g(t))$ is a RF whose curvature operator is positive (negative) initially then it remains so along the RF.